

# Fourier Transforms Revisited

- Consider the Fourier representation of a doubly infinitely long signal

$$X(j\omega_0) = \int_{-\infty}^{\infty} (e^{j\omega_0 t})^* x(t) dt$$

- This expression essentially answers the question

**"Is there a sinusoid  $e^{j\omega_0 t}$  in  $x(t)$  "**

# Fourier Transforms Revisited

Also it says

" There is a sinusoid  $e^{j\omega_0 t}$  of amplitude  $X(j\omega_0)d\omega_0$  in the infinitely long signal  $x(t)$  "

The implication is that  $x(t)$  can be written as a "sum" of weighted sinusoids

- Hence the inverse form that combines the sinusoids to give the infinite long signal is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega_0 t} X(j\omega_0) d\omega_0$$



# Fourier Transforms Revisited

- Sinusoids are
  - orthogonal
  - infinitely long
  - complete
  - deterministic
- **Orthogonality** means that a representation of a signal in terms of sinusoidal waveforms leads to a “least squares” problem, i.e. with a finite number of terms we have the best approximation of a given signal in the least square sense.

# Fourier Transforms Revisited

- **Orthogonality** also means that if we take a finite number of terms in an approximate representation, and **we wish to take one more** term then the previously calculated weighting coefficients remain unaffected.
- We need only calculate **the weight attached to the additional term**



# Fourier Transforms Revisited

- If we have **finite duration signals** which we try to represent in terms of **infinitely long sinusoids** we would expect to have problems.
- This is because the representation expects the signal to extend to infinity in both directions
- But it falls to zero outside its support, and at those points there are deviations (Gibbs ripples)



# Fourier Transforms Revisited

- We can think of the finite duration signal as **the product** of a top-hat window and an infinitely long signal
- Thus the spectrum of the infinitely long signal is “smeared” (see your PSD lectures) by the sinc function form of the spectrum of the top-hat window.
- One consequence is that one cannot discern the existence of a “line” in the spectrum of the given signal (see “uncertainty” shortly)



# Fourier Transforms Revisited

- **Completeness** means that if we take infinite number of terms in our representation then we would get an expression which is “identical” to the given signal
- Identity in this case means that in the least square sense nothing will be left behind ie the error becomes zero



# Fourier Transforms Revisited

- **Deterministic** means that we have an exact value of the signal at every point in time which may be got notionally from a formula
- Therefore it is not appropriate to use sinusoids, **as they are**, to represent finite duration stochastic, non-stationary signals (eg real world signals such as speech!)
- Either some modifications are to be made, or a new framework of representation to be devised



- If the given signal is of **finite bandwidth** (rather than finite duration) then similar observations can be made but now with respect to the **time-domain** behaviour
- The *Short-Time Fourier Transform* is a modification of the normal FT that responds to these needs

$$X_{ST}(j\omega_0, \tau) = \int_{-\infty}^{\infty} (e^{j\omega_0 t})^* g(t - \tau) x(t) dt$$

# STFT

- Where  $g(t)$  is a window shifted to the **temporal position** of interest
- Observe that now we have **lost orthogonality** and possibly completeness of representation
- The ideas of finite duration and bandwidth may be represented on a time-frequency figure as follows

# Uncertainty

- Principle of Uncertainty in Signals
- Consider a sinusoidal signal  $x[n] = \cos(n\theta_0)$  and take a finite segment in time, from  $-N$  to  $+N$
- Its spectrum is then

$$S(\theta) = \frac{\sin(2N+1)(\theta - \theta_0)/2)}{\sin((\theta - \theta_0)/2)}$$

# Uncertainty

- The main lobe is located between the frequencies

$$(2N + 1)(\theta_{1,2} - \theta_0) / 2 = \pm \pi$$

- Or 
$$\theta_{1,2} = \theta_0 \pm \frac{2\pi}{(2N + 1)}$$

- And hence its bandwidth is

$$BW = 2 \frac{2\pi}{(2N + 1)}$$

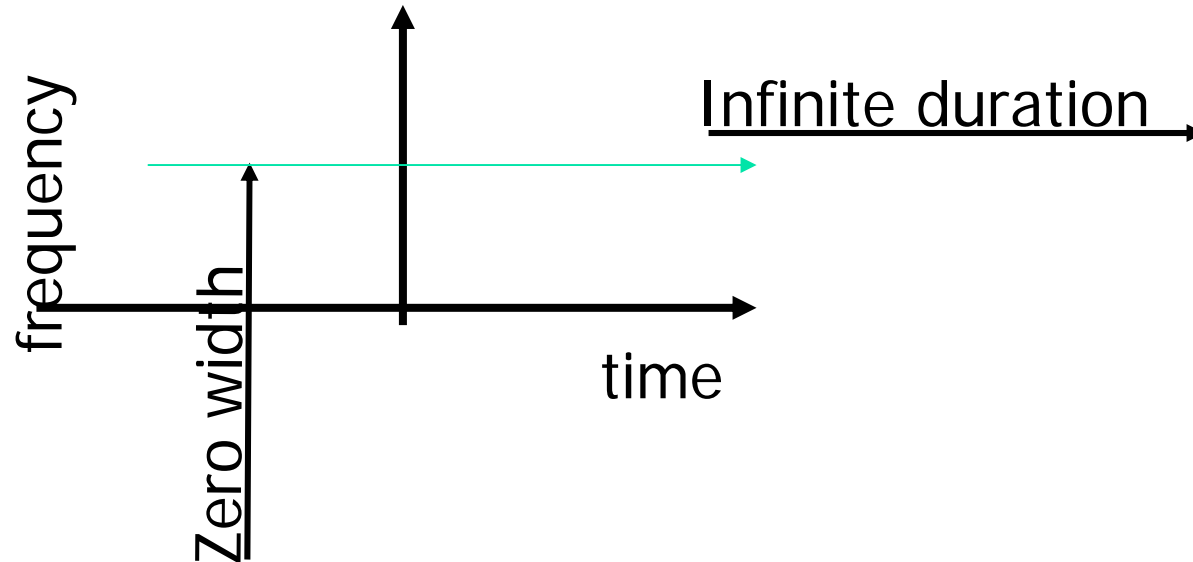


# Uncertainty

- It is clear that as the number of terms (i.e. duration of the signal) **increases** the bandwidth **decreases** in a reciprocal fashion
- Thus the frequency of the signal becomes **more precisely** defined as the signal length tends to infinity.

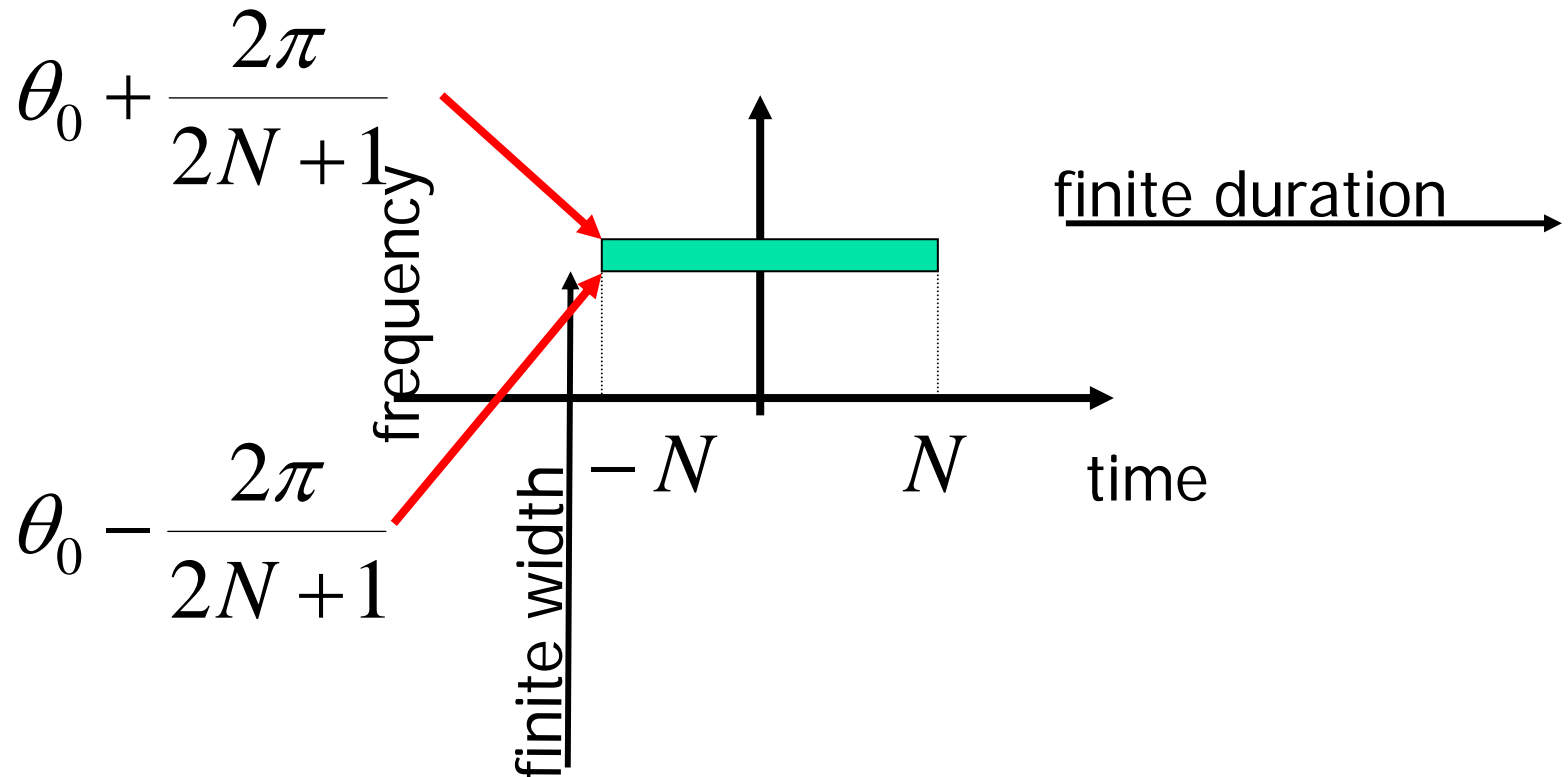
# Uncertainty

- We can represent these ideas on a time-frequency diagram.
- The ideal case (infinite duration)



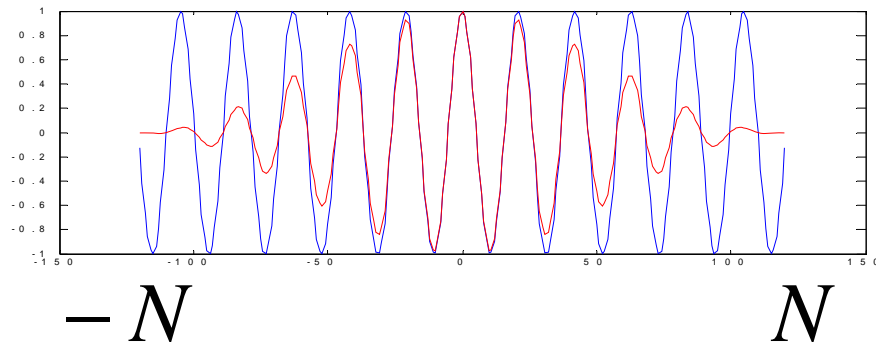
# Wavelets

- The finite bandwidth and duration case

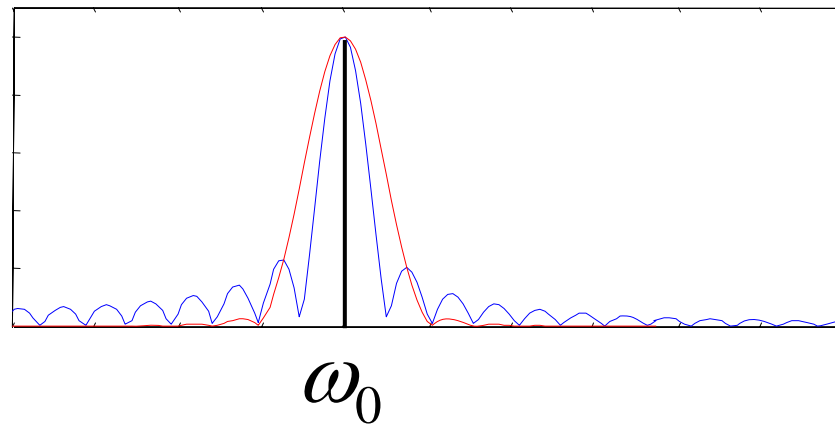


# Wavelets

$$g_{\omega_0}(t) = e^{j\omega_0 t} g(t)$$



time

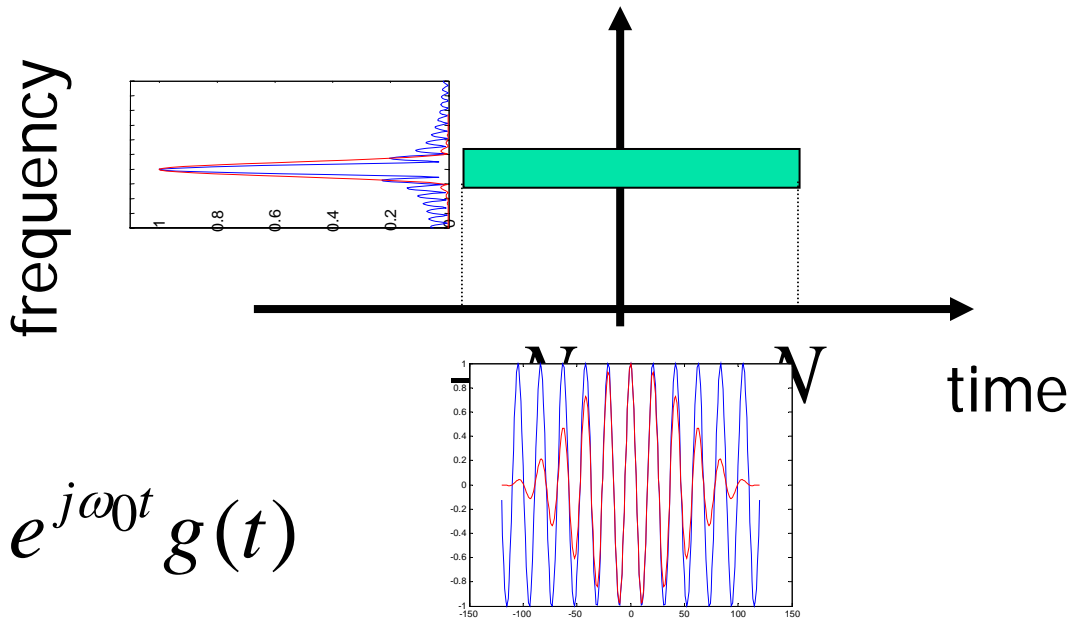


frequency



# Wavelets

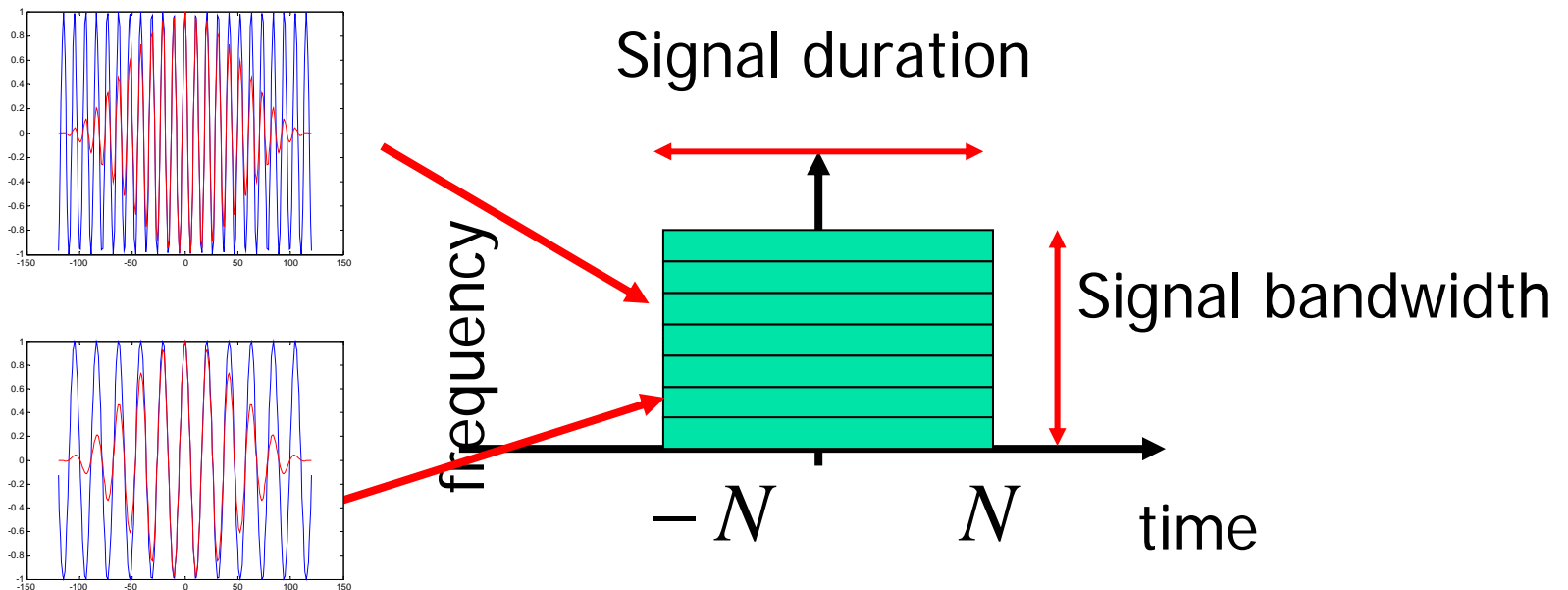
- If the DFT is used to analyse a signal the frequency resolution cannot be better than the DFT bin-width
- With **finite** length signals the diagram becomes



$$g_{\omega_0}(t) = e^{j\omega_0 t} g(t)$$

# Wavelets

- Thus if we have a general signal located in the region  $-N, +N$  we can “Fourier” (STFT) analyse using DFT.
- The time-frequency diagram would be





# Wavelets

- The “Fourier” analysis is now called the **Short Time Fourier Transform (STFT)** and it essentially asks the following question:

“Is there a short period sinusoidal wavefom of frequency  $\omega_0$  in the given signal and where is it located in time?”

- Clearly a filtering interpretation of the STFT is possible

# Degrees of Freedom

- Signals

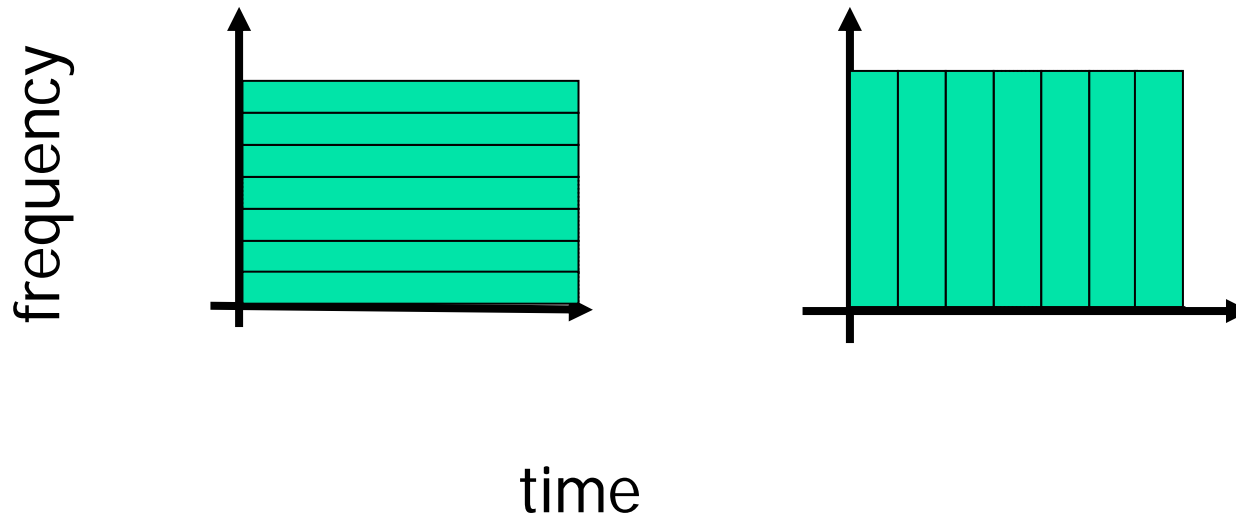
- bandlimited to  $\pm f_c$  and
- of finite duration  $\tau$

(These quantities need to be defined properly and are related to uncertainty!)

- Need  $2f_c\tau$  parameters for the complete description
- Thus the basic time-frequency rectangle can be divided in any way we desire so long as the total area is covered

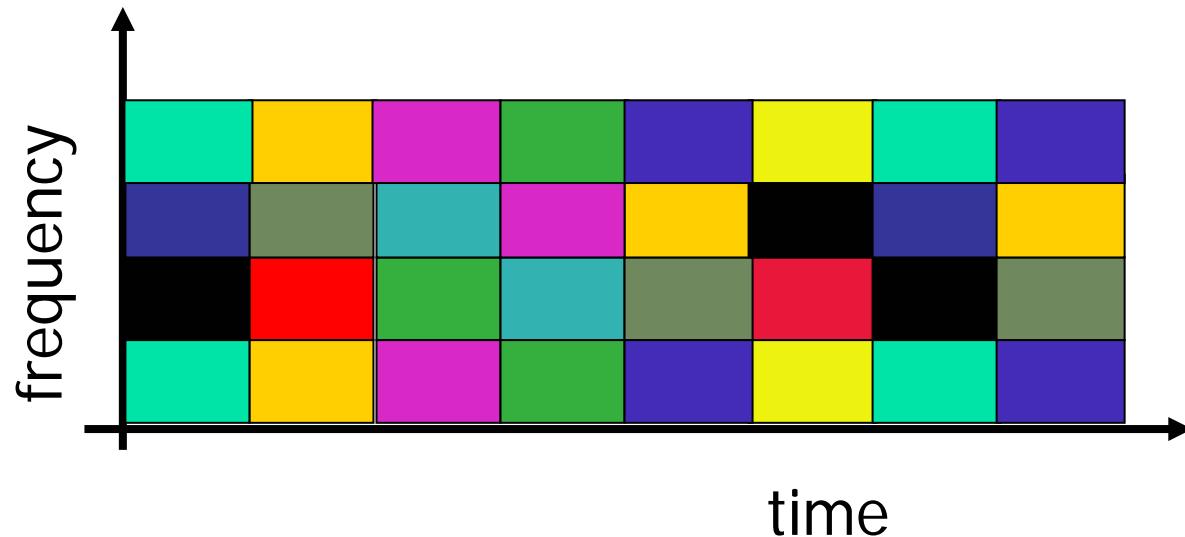
# Degrees of Freedom

- Possible partitions



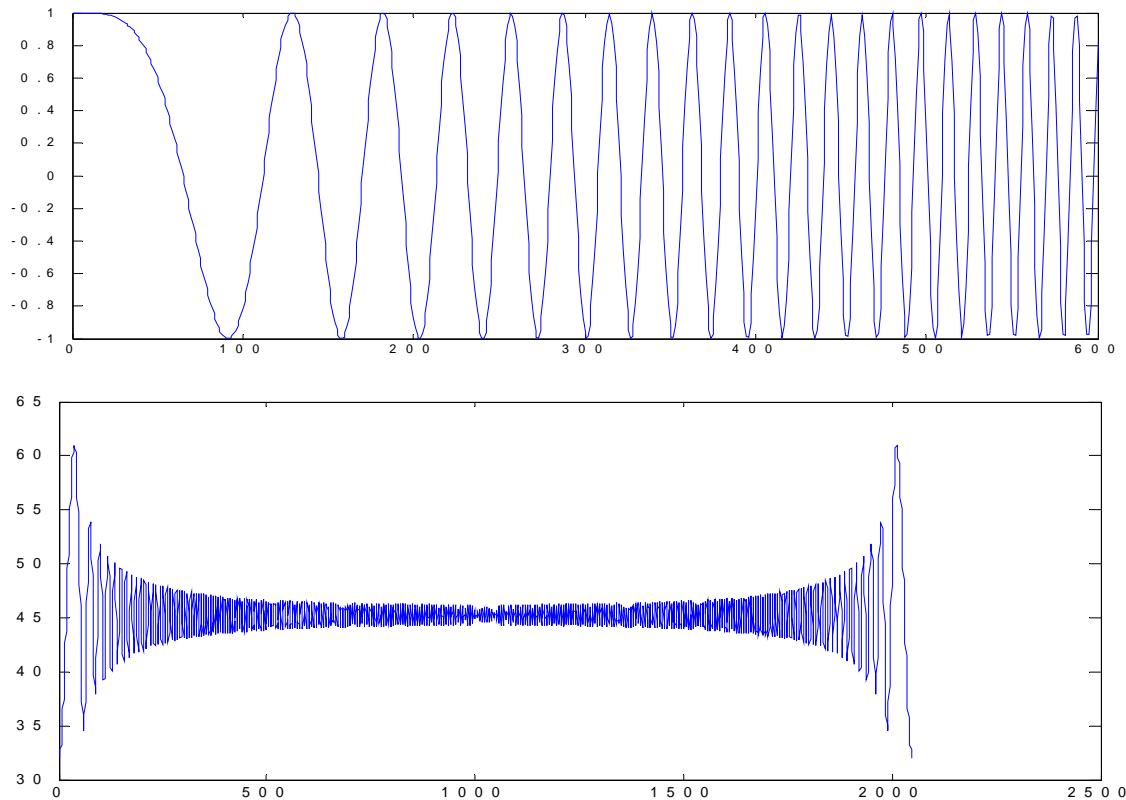
# Gabor Logons

- Other partitions are possible (Dennis Gabor, Imperial College 1946)



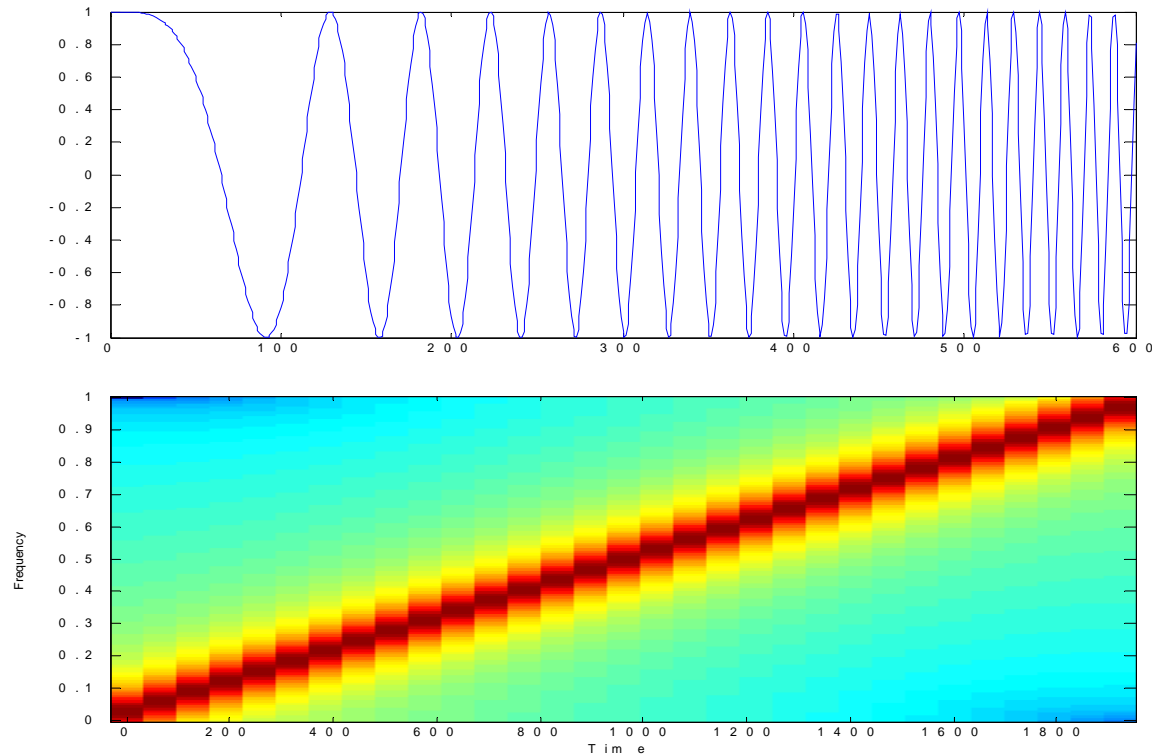
# Linear Chirp & FFT

- Take a linear chirp and its FFT



# Linear Chirp & STFT

- Linear chirp and STFT





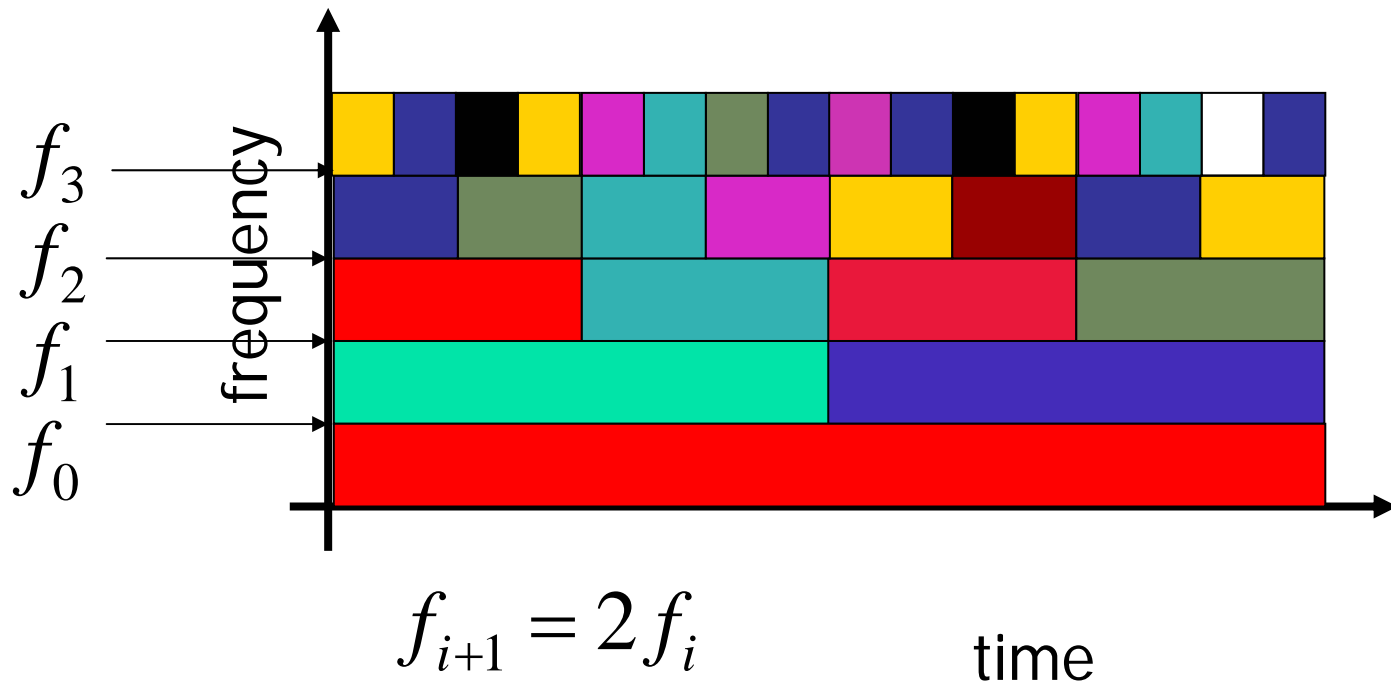


# Wavelets

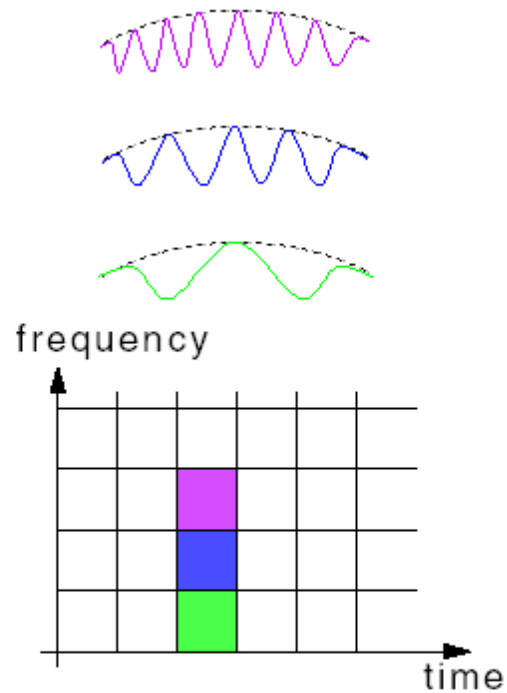
- In a frequency partitioning there is no need to retain **all** samples per band, as the sampling theorem is satisfied with fewer (in  $M$  frequency strips we need retain only 1 in  $M$  samples)
- Hence further partitions are possible

# Wavelets

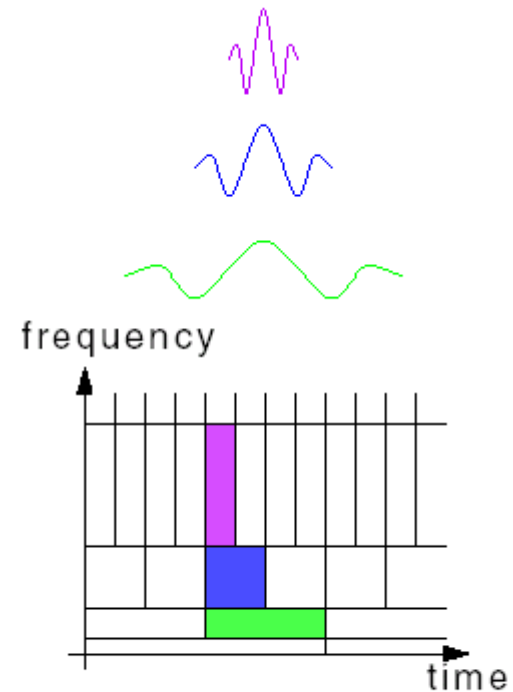
- A typical alternative is (others are possible!)



# STFT and Wavelets



short-time Fourier transform



wavelet transform



# STFT and Wavelets

- New partition recognises the following needs:
  - 1) At **high frequencies** we would like the time location of events in the signal to be made precise (ie longer data records)
  - 2) while at **low frequencies** we would like the frequency resolution to be more accurate (ie longer normalised bandwidths)



# Wavelets

- Contrive to select the new functions over the time-frequency support of each, to be orthogonal
- The mathematical form of the representation is very similar to the STFT

# Continuous Wavelet Transform

$$X_{WT}(\tau, a) = \frac{1}{\sqrt{|a|}} \int \Psi * \left( \frac{t - \tau}{a} \right) x(t) dt$$

- Compare with the STFT we looked at earlier

$$X_{ST}(j\omega_0, \tau) = \int_{-\infty}^{\infty} (e^{j\omega_0 t})^* g(t - \tau) x(t) dt$$

- The similarities are strikingly obvious!!!

# Continuous Wavelet Transform

$$X_{WT}(\tau, a) = \frac{1}{\sqrt{|a|}} \int x(t) \Psi * \left( \frac{t - \tau}{a} \right) dt$$

The transformed signal is a function of two variables:

- $\tau$  - *translation* parameter
- $a$  - *scale (or dilation)* parameter

$\Psi(t)$  is called the *mother wavelet*



# Continuous Wavelet Transform

• In a similar way we can think of the above expression as asking the question

“ Is there one these functions

$$\Psi \left( \frac{t - \tau}{a} \right)$$

in the given signal and where is it located?”.

• The expression is a convolution and this leads to a filtering interpretation





# How to Wavelet Transform

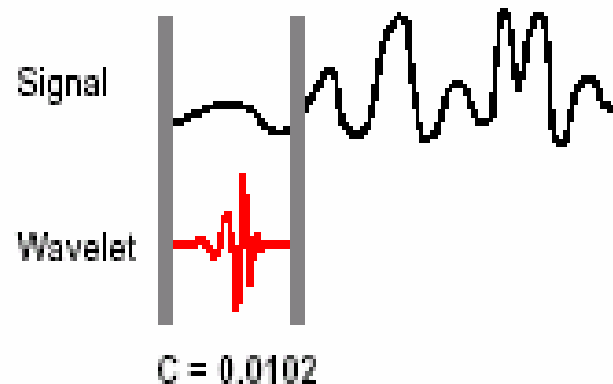
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## ■ Five Easy Steps to a Continuous Wavelet Transform

■ The continuous wavelet transform is the sum over all time of the signal multiplied by scaled, shifted versions of the wavelet. This process produces wavelet coefficients that are a function of scale and position.

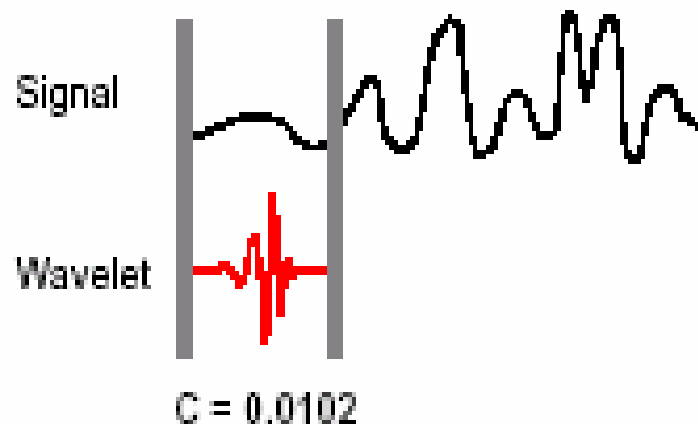
# How to Wavelet Transform

1 Take a wavelet and compare it to a section at the start of the original signal.



# How to Wavelet Transform

**2** Calculate a number,  $C$ , that represents how closely correlated the wavelet is with this section of the signal. The higher  $C$  is, the more the similarity. Note that the results will depend on the shape of the wavelet you choose.



# How to Wavelet Transform

**3** Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.

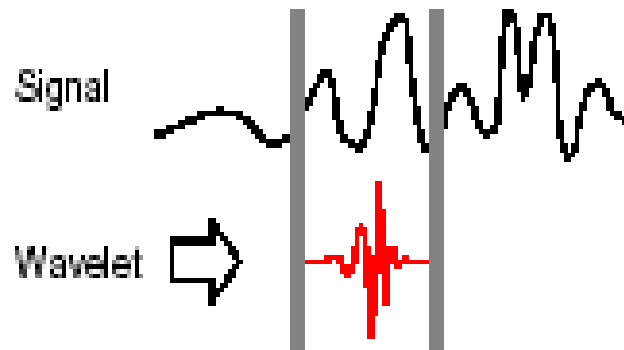
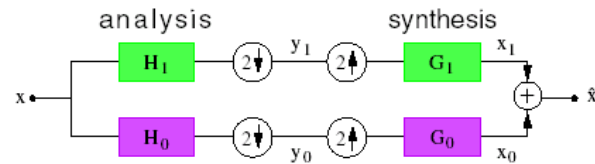


Figure 1 displays a plot of a signal (black line) and a wavelet (red line) over time. The signal is a complex waveform, and the wavelet is a localized pulse. The correlation coefficient  $C$  is 0.2247.

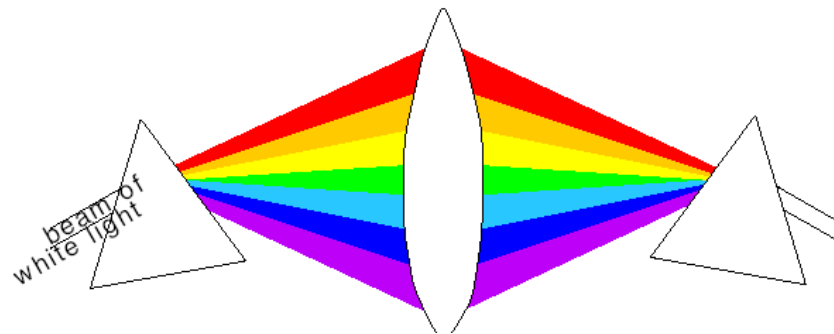
Professor A G Constantinides©

# Subband Coding Revisited

How do filter banks expand signals?

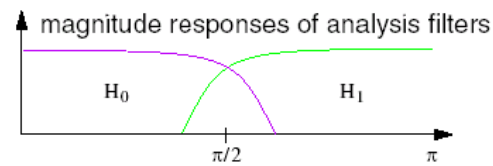
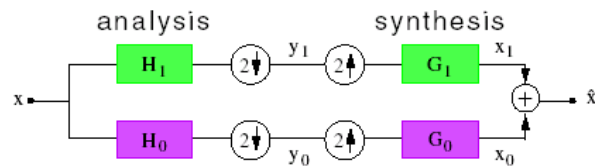


Analogy



# Subband Coding Revisited

## Perfect reconstruction filter banks

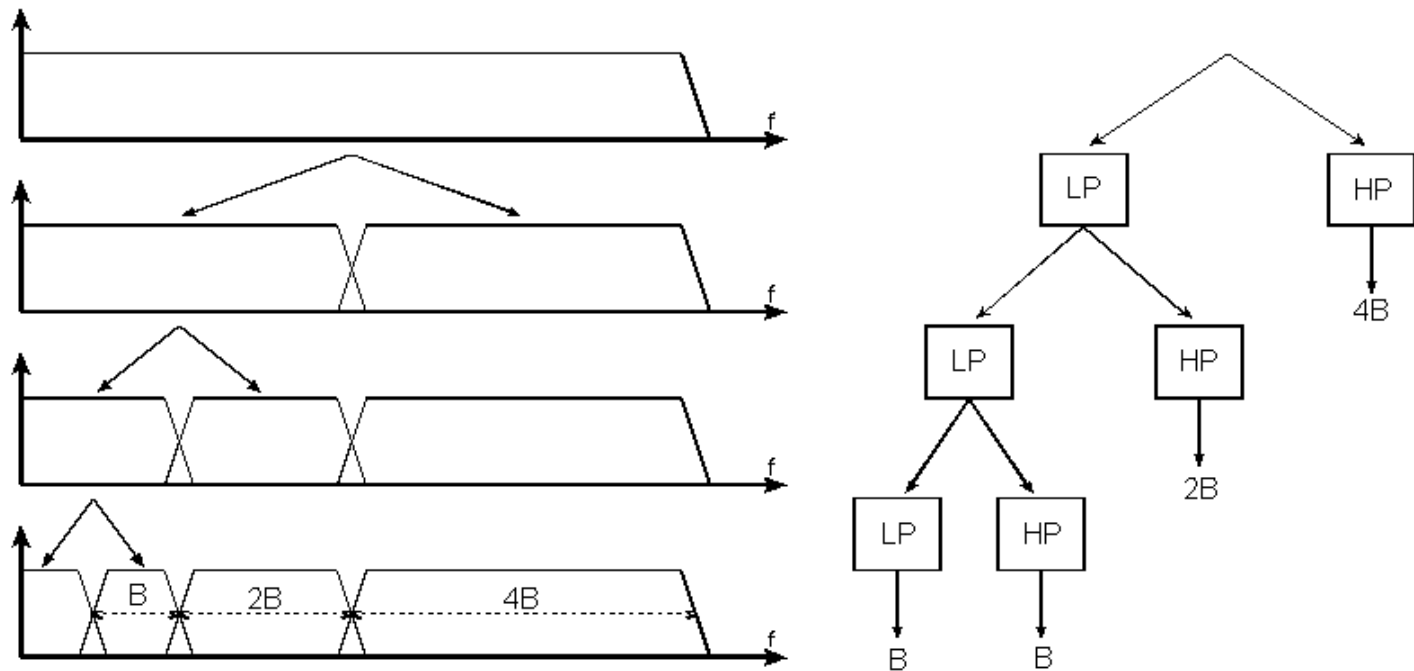


**Perfect reconstruction:**  $G_0 H_0 + G_1 H_1 = I$

**Orthogonal system:**  $(H_0)^* H_0 + (H_1)^* H_1 = I G_0 = (H_1)^*$

# Subband Coding

Split the signal spectrum with a bank of filters as:



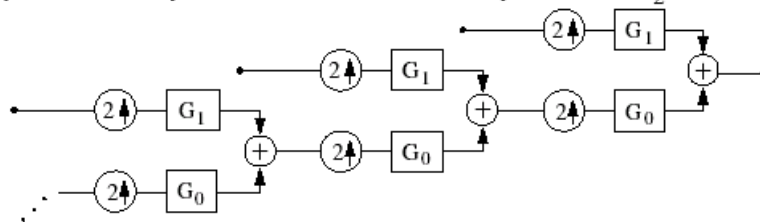


# Subband: Reconstruction

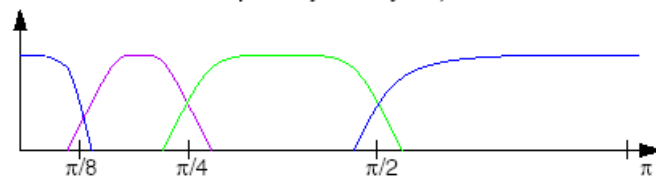
Split the signal spectrum with a bank of filters as:

Daubechies' construction...  
... iterated filter banks

Iteration will generate an orthonormal basis for the space of square-summable sequences  $l_2(\mathbb{Z})$



Consider equivalent basis sequences  $G_0^{(i)}(z)$  and  $G_1^{(i)}(z)$   
(generates octave-band frequency analysis)



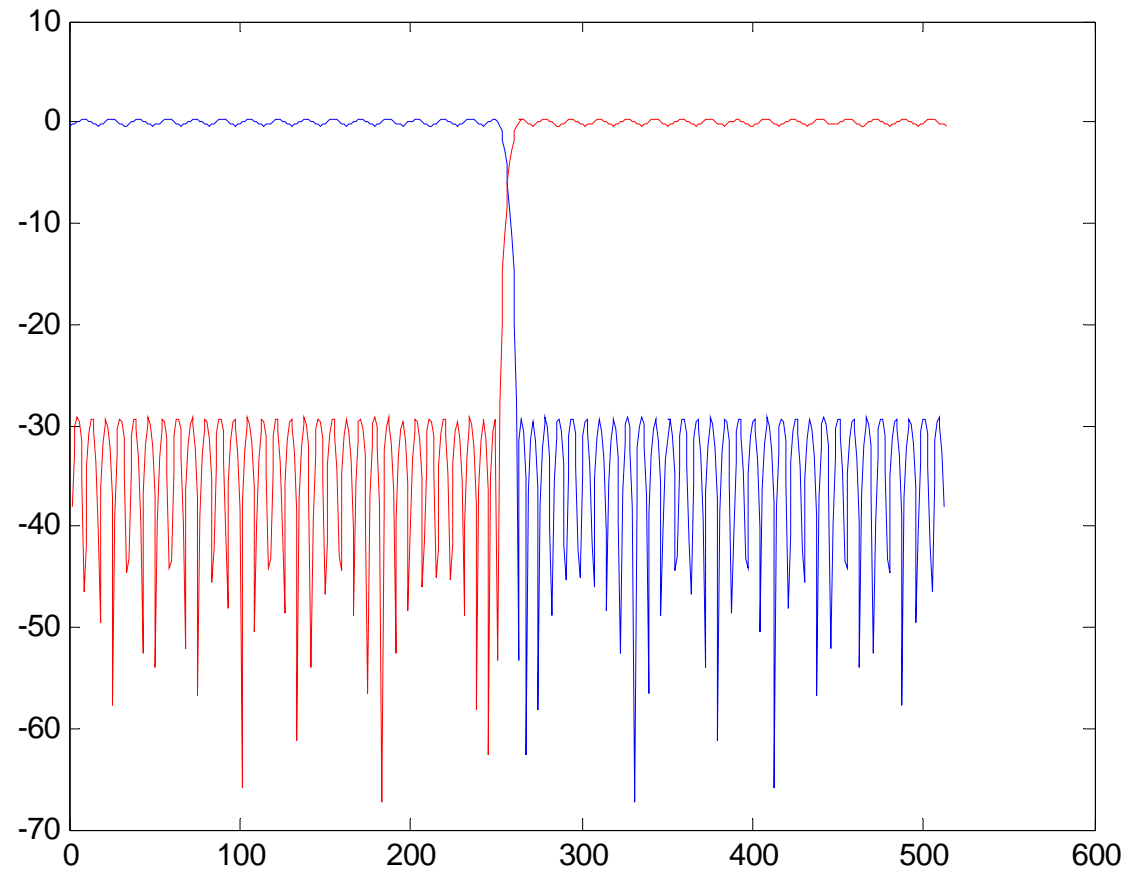
Interesting question: what happens in the limit?

# Subband Coding

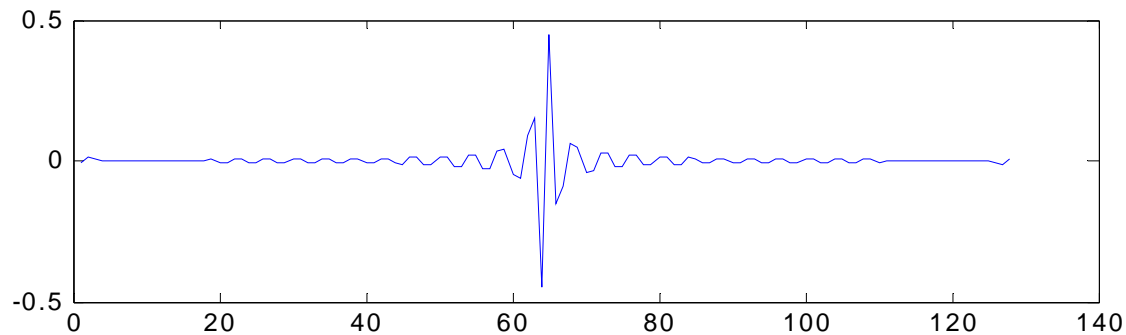
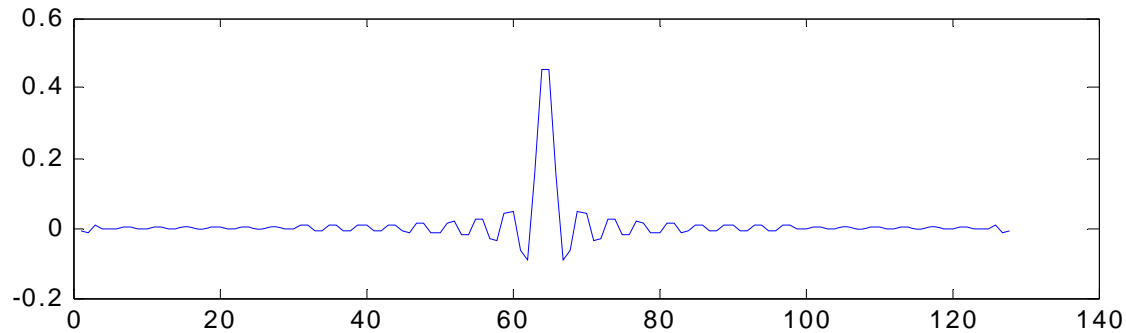
- 1) Arrange for the transfer function of  $h$  and for its  $p-1$  first derivatives to vanish at  $\omega=\pi$
- 2) Set  $g[n] = (-1)^n h[n]$ 

This is recognised as the special case of the lowpass to high pass digital filter transformation (Constantinides) for half-band filters
- 3) Arrange for  $g$  and  $h$  to be orthogonal
- 4) Perfect reconstruction conditions are satisfied (Vetterli)

# Subband Coding

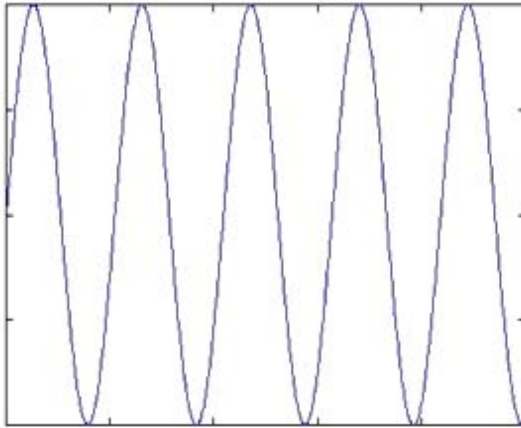


# Subband Coding



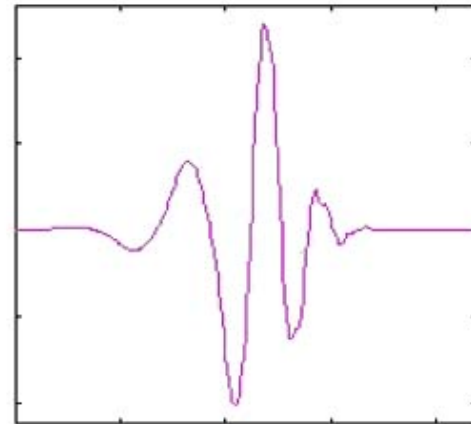
$$\mathbf{h}^T \mathbf{g} = -6.9999\text{e} - 018$$

# Sine waves and wavelets



**Sine wave**

Smooth  
Non-local (stretch out to infinity)

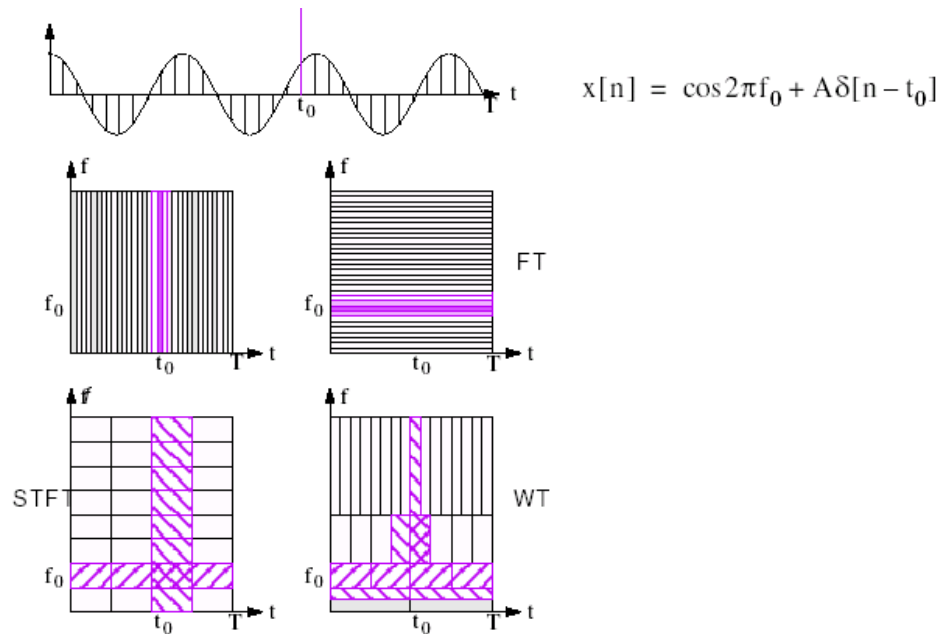


**Wavelet**

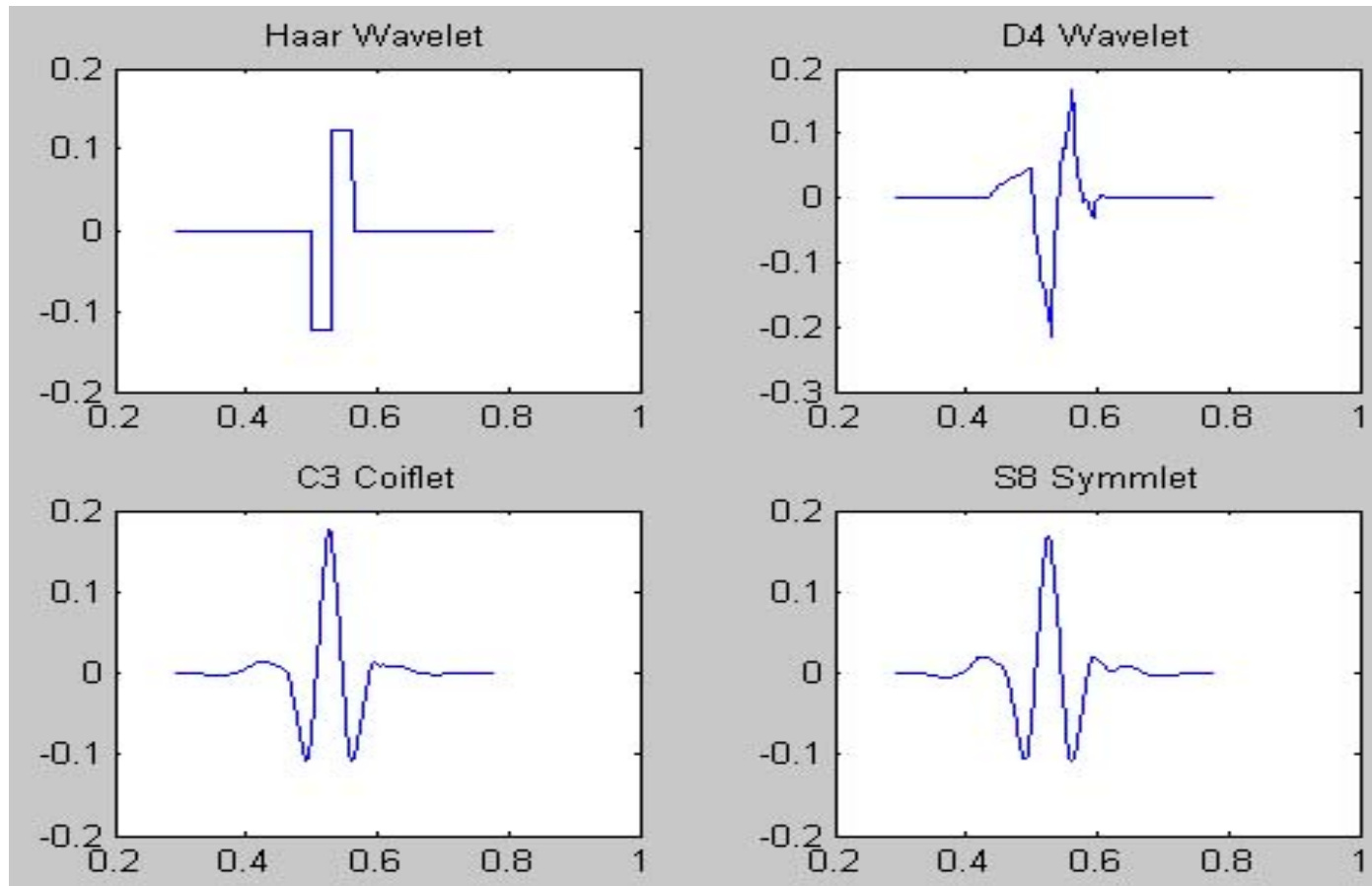
Irregular in shape  
Compactly supported  
(contained in finite domains)

# Fourier Transform and DWT

## ■ Example:



# Families of wavelets





# Applications of wavelets

Some areas of application :

1. *Image processing*

Increasing of quality image, image compression (wavelets are base of MPEG4)

2. *Signal processing :*

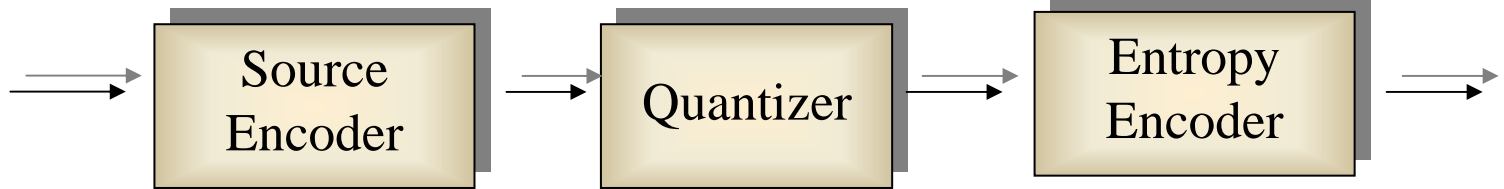
Noise reduction, compression, coding, analysis of non stationary data

*Other examples of wavelet applications are in astronomy, stock market, medicine, nuclear engineering, neurophysiology, music, optics etc.*



# Basic Wavelet Image Coder

Input  
Signal/Image

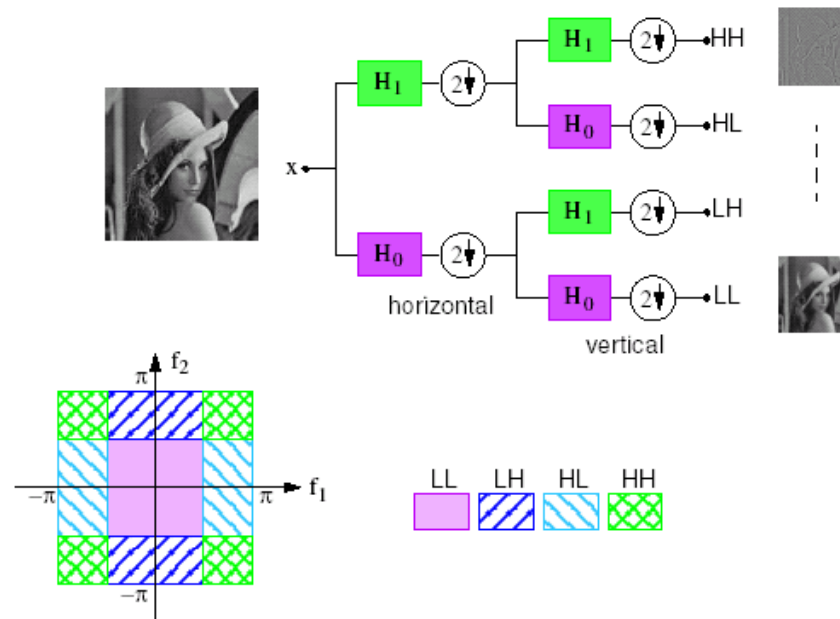


Compressed  
Signal/Image

There are three basic components in current wavelet coders:

- A decorrelating transform.
- A quantisation procedure
- An entropy coding procedure.

# Image Subband Coding



# Image Subbands

LL, LL	LH, LL	H,L
LL, LH	LH, LH	
L,H		H,H

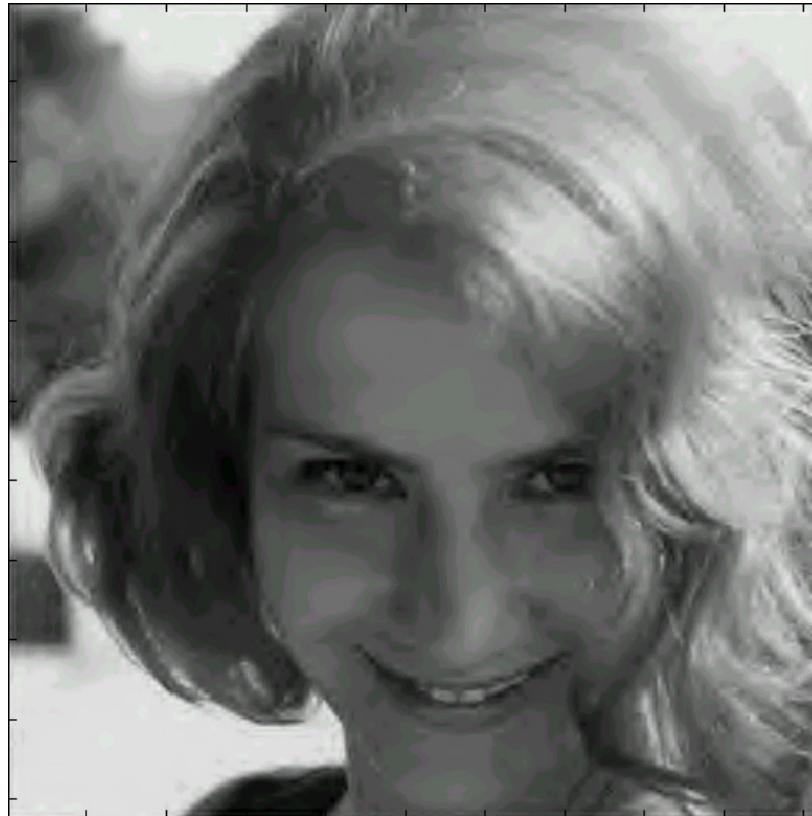
- The first letter corresponds to horizontal filtering, the last - to vertical
- L,H means, for example, that a lowpass is used in the first stage and a highpass in the second

# DWT vs. DCT



Original Image

# DWT vs. DCT



98% Wavelet Compression

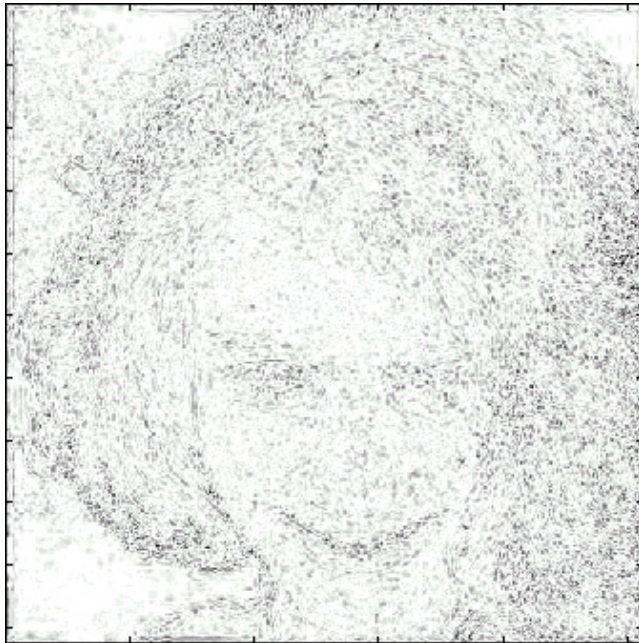
# DWT vs. DCT



98% DCT Compression



# Residual errors



98% Wavelet Compression



98% DCT Compression

# Denoising

- A noisy signal and its reconstruction. A threshold on the Wavelets coefficients has been imposed

